

Construction of Unrestricted-Rate Parallel Random Input-Output Code

Shan Lu, Horoshi Kamabe

Department of Electrical, Electronic,
and Computer Engineering,
Gifu University,
1-1, Yanagido, Gifu, 501-1193 Japan
Email: shan.lu.jp@ieee.org

Jun Cheng

Department of Intelligent Information
Engineering and Sciences,
Doshisha University,
Kyotanabe, Kyoto, 610-0321 Japan
Email: jcheng@ieee.org

Akira Yamawaki

Department of Electrical, Electronic,
and Computer Engineering,
Gifu University,
1-1, Yanagido, Gifu, 501-1193 Japan
Email: yamawaki@kmb.info.gifu-u.ac.jp

Abstract—A coding scheme for two-page unrestricted-rate P-RIO code that each page may have different code rates is proposed. In the second page, the code for each messages consists of two complementary codewords with code length n . There are a total of 2^{n-1} codes which are disjoint to guarantees uniquely-decodable for 2^{n-1} messages. In the first page, the code for each message consists of all weight- u vectors with their non-zero elements restricted to $(2u-1)$ same positions, where non-negative integer u is less than or equal to half of code length. Finding codes to be disjoint in first page is equivalent to construction of constant-weight codes, and the numbers of disjoint codes are the best-known numbers of codewords in constant-weight codes. Our coding scheme is constructive, and the code length is arbitrary. The sum rates of our proposed codes are higher than those of previous work.

I. INTRODUCTION

Flash memories are the prevalent type of non-volatile memory (NVM) in use today which are intended for SSD and mobile applications. Flash memories are comprised of blocks of cells. The cells can have binary values or multiple levels. Multilevel flash memories can store multiple bits in a cell. Conventionally, in multilevel flash memory, in order to read a single logical page, more than a single read threshold, on average, is required.

To increase input/output (I/O) performance for multilevel flash memories. A random input-output (RIO) code [1] is proposed. It is a coding scheme that permits writing $q-1$ pages in q levels and reading one page of data from multi-level flash memories only uses one single read threshold. A one-to-one correspondence between RIO codes [1] and the well studied WOM codes [3] is shown. However, in WOM codes, the encoder sets the cell state values based on the current memory state and the received message on each write, thus, the message are stored sequentially and are not all known in advance while encoding. Moreover, it is difficult to use difference code rate for each writing in WOM codes.

In RIO codes, all the messages of each level can be known in advance, it is unnecessary to store messages sequentially and possible to control difference code rate for each level. Therefore, a parallel RIO (P-RIO) code [2] is proposed that the encoding of each level is done in parallel while reading one page of data using a single read threshold. Higher sum-rates

of P-RIO codes are achieved [2] than that of RIO codes [1]. In [2], only fixed-rate P-RIO code is considered that the rates of each page are same. The complexity of research algorithm in [2] increases exponentially and the search space quickly becomes memory and computationally intensive as code length increases.

In this paper, we propose a coding scheme for two-page unrestricted-rate P-RIO code that each page may have different code rates. In the second page, the code for each messages consists of two complementary codewords with code length n . There are a total of 2^{n-1} codes which are disjoint to guarantees uniquely-decodable for 2^{n-1} messages. In the first page, the code for each message consists of all weight- u vectors with their non-zero elements restricted to $(2u-1)$ same positions, where non-negative integer u is less than or equal to half of code length. Finding codes to be disjoint in first page is equivalent to construction of constant-weight codes, and the numbers of disjoint codes are the best-known numbers of codewords in constant-weight codes. Our coding scheme is constructive, and the code length is arbitrary. The sum rates of our proposed codes are higher than those of conventional fixed-rate P-RIO codes in [2].

II. PRELIMINARY

A. Unrestricted-Rate P-RIO Code

Assume that in the flash memory, the cells have q levels and all cells are in level zero initially. It is only possible to increase the level of each cell. We denote by $[q] = \{0, 1, \dots, q-1\}$. For n cells, a vector $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in [q]^n$ will be called a cell state vector.

P-RIO code is a coding scheme that encoding of each page (level) in parallel while reading one page of data using a single read threshold. In a q -level flash memory, there are $(q-1)$'s pages. Let M_i , $i = 1, 2, \dots, q-1$, be the number of messages for i th page.

We expand the definition of P-RIO codes in [2] to $(n; M_1, M_2, \dots, M_{q-1})$ -P-RIO code as follows.

Definition 1 ([2]): An $(n; M_1, M_2, \dots, M_{q-1})$ -P-RIO code is a code with an encoding scheme comprising of n

cells with q levels and is defined by encoding map

$$\xi : [M_1] \times [M_2] \times \cdots \times [M_{q-1}] \rightarrow [q]^n$$

and decoding maps $\mathfrak{D}_i : [q]^n \rightarrow [M_i]$, for $i = 1, 2, \dots, q-1$, where messages of each page $(m_1, m_2, \dots, m_{q-1}) \in [M_1] \times [M_2] \times \cdots \times [M_{q-1}]$. \square

The sum rate of $(n; M_1, M_2, \dots, M_{q-1})$ -P-RIO code is

$$R_{\text{sum}} = \frac{\sum_{i=1}^{q-1} \log_2 M_i}{n}.$$

Note that the code rates of each page in $(n; M_1, M_2, \dots, M_{q-1})$ -P-RIO codes may be unequal, that is $M_i \neq M_j$ for $i \neq j$. In [2], only the P-RIO codes with same code rate of each page are considered.

The decoding maps $\mathfrak{D}_i : [q]^n \rightarrow [M_i]$ is first to read one page of data using a single read threshold, and then decode the data to corresponding message. Let r be a threshold level between pages $r-1$ and r , $r = 1, 2, \dots, q-1$, called r th threshold. The data of reading the r th threshold from the cell state $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in [q]^n$ is denoted as $\mathbf{d}_r(\mathbf{x}) = (d_0, d_1, \dots, d_{n-1}) \in \{0, 1\}^n$ where

$$d_r(x_i) = \begin{cases} 0, & x_i < r \\ 1, & x_i \geq r \end{cases}$$

Vector $\mathbf{d}_r(\mathbf{x})$ gives the $(q-r)$ th level's message.

B. $(n; M_1, M_2)$ -P-RIO Code

Now we consider $q = 3$ flash memory, that is to say, there are two pages on each cell.

Let $\mathbf{a} = (a_0, a_1, \dots, a_{n-1})$ and $\mathbf{b} = (b_0, b_1, \dots, b_{n-1})$ be two n -vectors. We say that vector \mathbf{a} is included in vector \mathbf{b} , denoted by $\mathbf{a} \leq \mathbf{b}$, if and only if $a_i \leq b_i$ for $i = 0, 1, \dots, n-1$.

Given two sets $A, B \subset \{0, 1\}^n$, if there exists at least one vector $\mathbf{a} \in A$ and at least one vector $\mathbf{b} \in B$ such that $\mathbf{a} \leq \mathbf{b}$, we say that set A is included in set B , denoted by $A \leq B$.

Let set $\mathbf{B} = \{B_0, B_1, \dots, B_{M_2-1}\}$ with $B_j \subseteq \{0, 1\}^n$, $0 \leq j \leq M_2-1$. We say that A is included in \mathbf{B} , denoted by $A \leq \mathbf{B}$, if $A \leq B_j$ for all $j = 0, 1, \dots, 2^{n-1}$.

The following theorem is given in [2] with $M_1 = M_2$.

Theorem 1 ([2]): An $(n; M_1, M_2)$ P-RIO code exists if and only if two sets $\mathbf{A} = \{A_0, A_1, \dots, A_{M_1-1}\}$ and $\mathbf{B} = \{B_0, B_1, \dots, B_{M_2-1}\}$ with $A_i, B_j \subseteq \{0, 1\}^n$, $0 \leq i \leq M_1-1$, $0 \leq j \leq M_2-1$ that satisfy the following conditions:

- 1) $A_i \cap A_{i'} = \emptyset$ and $B_j \cap B_{j'} = \emptyset$ for all $0 \leq i, i' \leq M_1-1$, $0 \leq j, j' \leq M_2-1$, $i \neq i'$, $j \neq j'$.
- 2) For any i , $i = 0, 1, \dots, M_1-1$, $A_i \leq \mathbf{B}$. \square

From Theorem 1, we have that set \mathbf{A} is the code of first page, subset A_i is the constituent code for the i th message. Similarly, \mathbf{B} is the code of second page, subset B_j is the constituent code for the j th message. The first condition in Theorem 1 guarantees the uniquely-decodable for each page's messages. The second condition guarantees reading one page of data using a single read threshold.

Example 1: Table I gives an example of $(3; 5, 4)$ P-RIO code. The columns correspond to the symbol value

of the first page. The sets of the code are $\mathbf{A} = \{\{000\}, \{001\}, \{010\}, \{100\}, \{011, 110, 101\}\}$ corresponding the messages 0 – 4 of first page.

The rows correspond to the symbol value of the second page. The sets of the codes are $\mathbf{B} = \{\{000, 111\}, \{001, 110\}, \{010, 101\}, \{100, 011\}\}$ corresponding the messages 0 – 3 of second page. The code rate of two sets are 0.773 and 0.667, respectively. The sum rate is $R_{\text{sum}} = 1.44$.

TABLE I
($n = 3; M_1 = 5, M_2 = 4$)-P-RIO CODE

	0	1	2	3	4
0	000	112	121	211	122
1	001	002	120	210	220
2	010	102	020	201	202
3	100	012	021	200	022

When the message of the first page is 1, the message of the second page is 2, based on the Table I, we have that the cell state is 102.

Threshold level between level 0 and 1 is 1, threshold level between level 1 and 2 is 2. $d_1(102) = 101$, $d_2(102) = 001$. Since $101 \in B_2$, $001 \in A_1$, thus, the message of the first page is 1, the message of the second page is 2. \square

III. CONSTRUCTION OF $(n; M_1, M_2)$ -P-RIO CODE

In this section, we give a construction of $(n; M_1, M_2)$ -P-RIO code.

A. Construction of \mathbf{B}

Let binary n -vector $\mathbf{b}_j = (b_0, b_1, b_2, \dots, b_{n-1})$ with its index $j = \sum_{l=0}^{n-2} 2^{n-1-l} b_l$, a decimal representation of \mathbf{b}_j . We define

$$B_j = \{\mathbf{b}_j, \bar{\mathbf{b}}_j = \mathbf{1} \oplus \mathbf{b}_j\}, j = 0, 1, \dots, 2^{n-1} - 1 \quad (1)$$

where $\mathbf{1}^n$ is all-1 n -vector and notation \oplus represents a modulo-2 sum of two binary vectors. Obviously, $B_j \cap B_{j'} = \emptyset$ for $j \neq j'$. Thus we obtain $\mathbf{B} = \{B_0, B_1, \dots, B_{M_2-1}\}$ with $M_2 = 2^{n-1}$.

Let $w(\mathbf{b})$ be the Hamming weight of vector \mathbf{b} . From (1) we have

$$w(\mathbf{b}_j) + w(\bar{\mathbf{b}}_j) = n. \quad (2)$$

Note that there always exists one vector in B_j such that its weight is greater than or equal to $\lceil \frac{n}{2} \rceil$, where $\lceil p \rceil$ is the smallest integer greater than or equal to p .

B. Construction of \mathbf{A}

In this section, for a given $0 \leq u \leq \lceil \frac{n}{2} \rceil$, we will show that a constant-weight set, consisting of all weight- u vectors with their non-zero elements restricted to $(2u-1)$ same positions, is included in set \mathbf{B} . The disjoint ones, among the constant-weight sets with all possible permutations of $(2u-1)$ non-zero positions on n positions for all $u = 0, 1, \dots, \lceil \frac{n}{2} \rceil$, forms set \mathbf{A} .

For a given positive integer n , let u be an integer with $0 \leq u \leq \lceil \frac{n}{2} \rceil$ and $m = 2u-1$. Let $A_{\text{core}}^{(u)}$ be a collection of all m -vectors with Hamming weight u . We arrange all the $t_c =$

$\binom{m}{u}$ vectors in set $A_{\text{core}}^{(u)}$ into $t_c \times m$ matrix $T_{\text{core}}^{(u)}$, called a core matrix.

From $T_{\text{core}}^{(u)}$, we give a $t_c \times n$ matrix

$$T^{(u,n)} = [\mathbf{t}_0^T, \mathbf{t}_1^T, \dots, \mathbf{t}_{m-1}^T, (\mathbf{0}^{t_c})^T, \dots, (\mathbf{0}^{t_c})^T] \quad (3)$$

where the most left m columns forms $T_{\text{core}}^{(u)}$, and the remaining $n - m$ columns \mathbf{t}_i^T , $m \leq i \leq n - 1$, are all-0 column vectors. Denote by $A^{(u,n)}$ a set of rows in matrix $T^{(u,n)}$.

Example 2: For $u = 2$, $m = 3$, we have $T_{\text{core}}^{(2)} = \begin{bmatrix} 110 \\ 011 \\ 101 \end{bmatrix}$,

$$T^{(2,4)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } A^{(2,4)} = \{1100, 0110, 1010\}.$$

□

Lemma 1: For $0 \leq u \leq \lceil \frac{n}{2} \rceil$, it follows that $A^{(u,n)} \leq \mathbf{B}$.

Proof: For any $\mathbf{a} \in A^{(u,n)}$, we rewrite $\mathbf{a} = (\mathbf{a}(0:m-1), \mathbf{0}^{n-m})$ where $\mathbf{a}(0:m-1) \triangleq (a_0, \dots, a_{m-1}) \in A_{\text{core}}^{(u)}$. From (1), for any j , it always satisfies that

$$w(\mathbf{b}_j(0:m-1)) + w(\bar{\mathbf{b}}_j(0:m-1)) = m = 2u - 1.$$

This implies that $w(\mathbf{b}_j(0:m-1)) \geq u$ or $w(\bar{\mathbf{b}}_j(0:m-1)) \geq u$.

Since $A_{\text{core}}^{(u)}$ includes all the m -vectors with weight u , there exists at least one, for example, $\mathbf{a}_{\ell_0}(0:m-1) \in A_{\text{core}}^{(u)}$, such that $\mathbf{a}_{\ell_0}(0:m-1) \leq \mathbf{b}_j(0:m-1)$ or $\mathbf{a}_{\ell_0}(0:m-1) \leq \bar{\mathbf{b}}_j(0:m-1)$. Therefore there exists at least one vector, $\mathbf{a}_{\ell_0} \in A^{(u,n)}$, such that $\mathbf{a}_{\ell_0} \leq \mathbf{b}_j$ or $\mathbf{a}_{\ell_0} \leq \bar{\mathbf{b}}_j$, for all $j = 1, 2, \dots, M_2$. This completes the proof. □

Let sequence $\Pi = (\pi_0, \pi_1, \dots, \pi_{n-1})$ is a permutation of the integers 0 to $n-1$. Denote a column permutation on $T^{(u,n)}$ by

$$T_{\Pi}^{(u,n)} = [\mathbf{t}_{\pi_0}^T, \mathbf{t}_{\pi_1}^T, \dots, \mathbf{t}_{\pi_{n-1}}^T]. \quad (4)$$

Let $A_{\Pi}^{(u,n)}$ be a set of rows in matrix $T_{\Pi}^{(u,n)}$. Similar to *Lemma 1*, we have the following lemma.

Lemma 2: For $0 \leq u \leq \lceil \frac{n}{2} \rceil$ and permutation Π , it follows that $A_{\Pi}^{(u,n)} \leq \mathbf{B}$. □

Since there are $(n - m)$'s all-zero columns in matrix $T_{\Pi}^{(u,n)}$, for all possible permutation, we have matrices $T_{\Pi_j}^{(u,n)}$ and their corresponding sets $A_{\Pi_j}^{(u,n)}$, $j = 0, 1, \dots, \binom{n}{m} - 1$. Let $\Pi_{\ell_0:\ell_{m-1}} = \{\pi_{\ell_0}, \pi_{\ell_1}, \dots, \pi_{\ell_{m-1}}\}$ be a set of non-zero columns' indexes in matrix $T_{\Pi}^{(u,n)}$. Denote by $|\cdot|$ a cardinality of set. The following lemma shows sufficient and necessary conditions for two permutation patterns such that their corresponding sets are disjoint.

Lemma 3: Two sets $A_{\Pi}^{(u,n)} \cap A_{\Pi'}^{(u,n)} = \emptyset$, if and only if

$$|\Pi_{\ell_0:\ell_{m-1}} \cap \Pi'_{\ell_0:\ell_{m-1}}| \leq u - 1. \quad (5)$$

Proof: (Sufficient condition:) Assume that $|\Pi_{\ell_0:\ell_{m-1}} \cap \Pi'_{\ell_0:\ell_{m-1}}| \leq u - 1$. This means $\Pi_{\ell_0:\ell_{m-1}}$ and $\Pi'_{\ell_0:\ell_{m-1}}$ have at most $u - 1$ same elements. Since every $\mathbf{a} \in A_{\Pi}^{(u,n)}$ and every $\mathbf{a}' \in A_{\Pi'}^{(u,n)}$ have weight of u , it follows $\mathbf{a} \neq \mathbf{a}'$. Thus $A_{\Pi}^{(u,n)} \cap A_{\Pi'}^{(u,n)} = \emptyset$.

(Necessary condition:) Assume that $A_{\Pi}^{(u,n)} \cap A_{\Pi'}^{(u,n)} = \emptyset$. Since weights of any vectors in these two sets are u , there exist at least u different elements between $\Pi_{\ell_0:\ell_{m-1}}$ and $\Pi'_{\ell_0:\ell_{m-1}}$. Thus $|\Pi_{\ell_0:\ell_{m-1}} \cap \Pi'_{\ell_0:\ell_{m-1}}| \leq u - 1$. This completes the proof. □

Lemma 3 means that among all possible $\binom{n}{m}$ permutations Π , the permutation patterns satisfying (5) gives disjoint sets. In practical, finding these permutations is not easy. Fortunately, we can find a solution in the related research field on constant-weight codes [4]. For a given codeword in a constant-weight code with code length n , minimum Hamming distance $2u$, and weight $m = 2u - 1$, the non-zero positions in the codeword are the non-zero columns' indexes in matrix $T_{\Pi}^{(u,n)}$. Let $M^{(u,n)}$ be the number of constant-weight codewords. The $M^{(u,n)}$ constant-weight codewords give $M^{(u,n)}$ permutation patterns and thus provide $\{A_0^{(u,n)}, A_1^{(u,n)}, A_{M^{(u,n)}-1}^{(u,n)}\}$ with $A_k^{(u,n)} \cap A_{k'}^{(u,n)} = \emptyset, k \neq k'$. Here we use the same notation for $A_{\Pi_j}^{(u,n)}$ and $A_k^{(u,n)}$ for convenience.

The permutation patterns satisfying (5) and the number of $M^{(u,n)}$ are given in Table II for $n \leq 15$ [4]–[6]. Note that $M^{(0,n)} = 1$ for $A_0^{(0,n)} = \{\mathbf{0}^n\}$.

Tables III and IV give all possible permutation patterns of $\Pi_{\ell_0:\ell_{m-1}}$ for forming disjoint sets.

TABLE II
THE NUMBER OF $M^{(u,n)}$

$n \backslash u$	1	2	3	4	5	6	7
4	4	1	–	–	–	–	–
5	5	2	1	–	–	–	–
6	6	4	1	–	–	–	–
7	7	7	1	1	–	–	–
8	8	8	2	1	–	–	–
9	9	12	3	1	1	–	–
10	10	13	6	1	1	–	–
11	11	17	11	2	1	1	–
12	12	20	12	3	1	1	–
13	13	26	18	4	1	1	1
14	14	28	28	8	2	1	1
15	15	35	42	15	3	1	1

Moreover, for two distinct u and u' , it follows that $A_k^{(u,n)} \cap A_{k'}^{(u',n)} = \emptyset$ since the weights in these two sets are distinct. Let

$$\mathbf{A} = \{A_k^{(u,n)} | k = 0, 1, \dots, M^{(u,n)} - 1, u = 0, 1, \dots, \lceil \frac{n}{2} \rceil\}. \quad (6)$$

We have the main result of this work.

Theorem 2: Sets \mathbf{A} in (6) and \mathbf{B} in (1) form an

$$(n; M_1, M_2 = 2^{n-1})\text{P-RIO code},$$

where $M_1 = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} M^{(i,n)}$. □

Let us look more closely at $A_k^{(u=\lceil \frac{n}{2} \rceil, n)}$, $k = 0, 1, \dots, M^{(u,n)} - 1$ in (6). In the case of odd n , there does not exist any all-zero column in $T^{(\lceil \frac{n}{2} \rceil, n)}$ of (3). Thus sets $A_{\Pi}^{(\lceil \frac{n}{2} \rceil, n)}$ from any possible permutations are the same as $A^{(\lceil \frac{n}{2} \rceil, n)}$ itself, and thus $M^{(\lceil \frac{n}{2} \rceil, n)} = 1$. Furthermore, we observe that

TABLE III
ALL POSSIBLE PATTERNS OF $\Pi_{\ell_0:\ell_{m-1}}$ FORMING DISJOINT SETS ($u = 2$)

n	$M^{(2,n)}$	$\Pi_{\ell_0:\ell_{m-1}}(u = 2)$
5	2	$\{0, 2, 3\}, \{1, 3, 4\}$
6	4	$\{0, 1, 2\}, \{0, 3, 4\}, \{1, 3, 5\}, \{2, 4, 5\}$
7	7	$\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}$
8	8	$\{0, 1, 2\}, \{5, 6, 7\}, \{0, 3, 5\}, \{1, 3, 6\}, \{2, 3, 7\}, \{0, 4, 6\}, \{1, 4, 7\}, \{2, 4, 5\}$
9	12	$\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}, \{0, 4, 8\}, \{1, 5, 6\}, \{2, 3, 7\}, \{0, 5, 7\}, \{1, 3, 8\}, \{2, 4, 6\}$
15	35	$\{3, 7, 11\}, \{3, 8, 12\}, \{3, 9, 13\}, \{3, 10, 14\}, \{4, 7, 12\}, \{4, 8, 13\}, \{4, 9, 14\}, \{4, 10, 11\}, \{5, 7, 13\}, \{5, 8, 14\}, \{5, 9, 11\}, \{5, 10, 12\}, \{6, 7, 14\}, \{6, 8, 11\}, \{6, 9, 12\}, \{6, 10, 13\}$

TABLE IV
ALL POSSIBLE PATTERNS OF $\Pi_{\ell_0:\ell_{m-1}}$ FORMING DISJOINT SETS ($u = 3$)

n	$M^{(3,n)}$	$\Pi_{\ell_0:\ell_{m-1}}(u = 3)$
7	1	$\{0, 1, 2, 3, 4\}$
8	2	$\{0, 1, 2, 3, 4\}, \{0, 1, 5, 6, 7\}$
9	3	$\{0, 1, 2, 3, 4\}, \{3, 4, 5, 6, 7\}, \{0, 1, 6, 7, 8\}$
10	6	$\{0, 1, 2, 3, 4\}, \{0, 1, 5, 6, 7\}, \{1, 2, 5, 8, 9\}, \{2, 3, 6, 7, 8\}, \{3, 4, 5, 6, 9\}, \{0, 4, 7, 8, 9\}$

$A^{(\lceil \frac{n}{2} \rceil, n)} = E^{(\lceil \frac{n}{2} \rceil, n)}$, where $E^{(u,n)}$ is the set of all n -vector with Hamming weight u .

In the case of even n , there exists one all-zero column in $T^{(\frac{n}{2}, n)}$ of (3), and thus n possible permutation patterns gives n possible matrices, $T_{\Pi}^{(\frac{n}{2}, n)}$ s. According to Lemma 3, the corresponding n possible sets, $A_{\Pi}^{(\frac{n}{2}, n)}$ s, are joint each other. As a result, we choose one, e.g., $A^{(\frac{n}{2}, n)}$, among $A_{\Pi}^{(\frac{n}{2}, n)}$ s, in our proposed code in Theorem 2, and thus $M^{(\frac{n}{2}, n)} = 1$. Different from the case of odd n , we observe that $A^{(\frac{n}{2}, n)} \subset E^{(\frac{n}{2}, n)}$ since $A^{(\frac{n}{2}, n)}$ consists of only the weigh- $\frac{n}{2}$ vectors with the most right bit being 0. This observation motives us to improve our proposed codes in Theorem 2 by adding a supplemental set to A .

Specifically, given an even n , let $\bar{A}^{(\frac{n}{2}, n)} = \{1 \oplus a | a \in A^{(\frac{n}{2}, n)}\}$ and $A_{\text{sup}} = \bar{A}^{(\frac{n}{2}, n)} \cup E^{(\frac{n}{2}+1, n)}$. Adding A_{sup} to A of (6) provides $A' = \{A, A_{\text{sup}}\}$.

Theorem 3: For a given even n , sets A' and B in (1) form an

$$(n; M'_1, M_2 = 2^{n-1}) \text{ P-RIO code,}$$

where $M'_1 = M_1 + 1$.

Proof: We first show that $A_{\text{sup}} \leq B$. We partition B into two parts $B^{\frac{n}{2}}$ and $\bar{B}^{\frac{n}{2}}$. The first part $B^{\frac{n}{2}}$ consists of B_j s whose elements are with weight $n/2$. The remaining B_j s form $\bar{B}^{\frac{n}{2}}$. Since $w(b_j) + w(\bar{b}_j) = n$, every $B_j \in \bar{B}^{\frac{n}{2}}$ has a vector whose weight is greater than $n/2 + 1$. Therefore we have $E^{(\frac{n}{2}+1, n)} \leq \bar{B}^{\frac{n}{2}}$.

We now show $\bar{A}^{(\frac{n}{2}, n)} < B^{\frac{n}{2}}$. Among $\bar{A}^{(\frac{n}{2}, n)}$, the most right bit of every vector is 1, and the remaining bits forms the all subvectors with weight $n/2 - 1$. In $B^{\frac{n}{2}}$, one of two vectors always has the most right bit being 1 and remaining bits be weight $n/2 - 1$. Therefore $\bar{A}^{(\frac{n}{2}, n)} \leq B^{\frac{n}{2}}$. It follows that $A_{\text{sup}} \leq B$.

It remains to show $A_{\text{sup}} \cap A = \emptyset$. Since the most right bit in $\bar{A}^{(\frac{n}{2}, n)}$ is 1 while that in $A^{(\frac{n}{2}, n)}$ is 0, it follows that $\bar{A}^{(\frac{n}{2}, n)} \cap A^{(\frac{n}{2}, n)} = \emptyset$ and thus $\bar{A}^{(\frac{n}{2}, n)} \cap A = \emptyset$. Also we have $E^{(\frac{n}{2}+1, n)} \cap A = \emptyset$ because weights in the two sets are distinct. Therefore $A_{\text{sup}} \cap A = \emptyset$. This completes the proof. \square

C. Code Rate of Two-Write Unrestricted-Rate P-RIO-Code

Based on $M^{(u,n)}$ in Table II, we have R_{sum} of $(n; M_1, M_2)$ P-RIO codes in Table V, compared with that of $(n; M, M)$ fixed-rate P-RIO code [2]. From the table, we see that the sum rates of our P-RIO codes are higher than those of the fixed P-RIO codes given in [2] when $n = 3, 4, 5, 6$. When $n > 6$, there is no data in [2] since construction complexity is very high.

TABLE V
 $(n; M_1, M_2; 2)$ P-RIO CODES FOR $3 \leq n \leq 15$

n	M_1	M_2	R_{sum}	M [2]	R'_{sum} [2]
3	5	4	1.44	4	1.333
4	7	8	1.452	7	1.4037
5	9	16	1.434	11	1.384
6	13	32	1.45	19	1.41
7	17	64	1.44	/	/
8	21	128	1.424	/	/
9	27	256	1.417	/	/
10	33	512	1.404	/	/
11	44	1024	1.406	/	/
12	51	2048	1.389	/	/
13	65	4096	1.386	/	/
14	84	8192	1.385	/	/
15	114	16384	1.389	/	/

D. Examples

Now we give two examples of P-RIO codes with code length of $n = 4, 5$.

Example 3: For $n = 4$, by (1), we have the set of second page of

$$B = \{\{0000, 1111\}, \{0001, 1110\}, \{0010, 1101\}, \{0011, 1100\}, \{0100, 1011\}, \{0101, 1010\}, \{0110, 1001\}, \{0111, 1000\}\},$$

Next, we construct A' as follows. When $u = 0$, $T_{\text{core}}^{(1)} = [0]$, and then $A_0^{(0,4)} = \{0000\}$. When $u = 1$, $T_{\text{core}}^{(1)} = [1]$,

TABLE VI
(5;9,16)-P-RIO CODE

	0	1	2	3	4	5	6	7	8
0	00000	21111	12111	11112	11121	11211	21211	12121	22211
1	00001	21110	12110	00002	11120	11210	21210	12120	22210
2	00010	21101	12101	11102	00020	11201	21201	12102	22201
3	00011	21100	12100	00012	00021	11200	21200	00022	22200
4	00100	21011	12011	11012	11021	00200	21021	12012	22021
5	00101	21010	12010	00102	11020	00201	21020	12020	22020
6	00110	21001	12001	11002	00120	00210	00220	12002	22002
7	00111	21000	12000	00112	00121	00211	00221	00122	00222
8	01000	20111	02000	10112	10121	10211	20121	10122	10222
9	01001	20110	02001	01002	10120	10210	20120	02002	20220
10	01010	20101	02010	10102	01020	10201	20201	02020	20202
11	01011	20100	02011	01012	01021	10200	20200	02012	02022
12	01100	20011	02100	10012	10021	01200	20021	10022	20022
13	01101	20010	02101	01102	10020	01201	20020	02102	02202
14	01110	20001	02110	10002	01120	01210	01220	02120	02220
15	01111	20000	02111	01112	01121	01211	01221	02112	01222

we have that $A_0^{(1,4)} = \{1000\}$, $A_1^{(1,4)} = \{0100\}$, $A_2^{(1,4)} = \{0010\}$, $A_3^{(1,4)} = \{0001\}$. When $u = 2$, from Example 2 we have $A_0^{(2,4)} = \{1100, 0110, 1010\}$.

Since n is even, $\bar{A}^{(2,4)} = \{0011, 1001, 0101\}$ and $E^{(3,4)} = \{1110, 1101, 1011, 0111\}$, from Theorem 3, we have $A_{\text{sup}} = \{0011, 1001, 0101, 1110, 1101, 1011, 0111\}$.

Therefore, we have $\mathbf{A}' = \{A_0^{(0,4)}, A_0^{(1,4)}, A_1^{(1,4)}, A_2^{(1,4)}, A_3^{(1,4)}, A_0^{(2,4)}, A_{\text{sup}}\}$. From Theorem 3, \mathbf{A}' and \mathbf{B} provide (4;7,8)-P-RIO code as shown in Table. VII. \square

TABLE VII
(4;7,8)-P-RIO CODE

	0	1	2	3	4	5	6
0	0000	2111	1211	1121	1112	2200	1122
1	0001	2110	1210	1120	0002	2210	2220
2	0010	2101	1201	0021	1102	2201	2202
3	0011	2100	1200	0021	0012	2200	0022
4	0100	2011	0200	1021	1012	2021	2022
5	0101	2010	0201	1020	0102	2020	0202
6	0110	2001	0210	0120	1002	0220	2002
7	1000	2000	0211	0121	0112	0221	0222

Example 4: For $n = 5$, from (1), we have

$$\mathbf{B} = \{\{00000, 11111\}, \{00001, 11110\}, \{00010, 11101\}, \{00011, 11100\}, \{00100, 11011\}, \{00101, 11010\}, \{00110, 11001\}, \{00111, 11000\}, \{01000, 10111\}, \{01001, 10110\}, \{01010, 10101\}, \{01011, 10100\}, \{01100, 10011\}, \{01101, 10010\}, \{01110, 10001\}, \{01111, 10000\}\}$$

When $u = 0$ and $u = 1$, we have $A_0^{(0,5)} = \{00000\}$, and $A_0^{(1,5)} = \{00001\}$, $A_1^{(1,5)} = \{00010\}$, $A_2^{(1,5)} = \{00100\}$, $A_3^{(1,5)} = \{01000\}$, $A_4^{(1,5)} = \{10000\}$.

When $u = 2$, from Table III, we have $A_0^{(2,5)} = \{10010, 10100, 10100\}$ and $A_1^{(2,5)} = \{01010, 01001, 00011\}$.

When $u = 3$, we have $A_0^{(3,5)} = \{00111, 01110, 11100, 01011, 10110, 01101, 11010, 10011, 11001, 10101\}$.

Finally, we obtain the (5;9,16)-P-RIO code in Table. VI.

IV. CONCLUSION

In this paper, we proposed a coding scheme for two-page unrestricted-rate P-RIO code that each page may have different code rates. Our coding scheme is constructive, and the code length is arbitrary. The sum rates of our proposed codes are higher than those of conventional fixed-rate P-RIO codes in [2].

ACKNOWLEDGEMENT

This work was supported by the Japan Society for the Promotion of Science through the Grant-in-Aid for Scientific Research (C) under Grant 16K06373, in part by MEXT through the Strategic Research Foundation at Private Universities (2014-2018) under Grant S1411030, and the Advanced Storage Research Consortium and JSPS KAKENHI Grant 15K00010.

REFERENCES

- [1] E. Sharon and I. Alrod, "Coding scheme for optimizing random I/O Performance," in *Proc. Non-Volatile Memories Workshop*, San Diego, CA, USA, Apr. 2013.
- [2] E. Yaakobi and R. Motwani, "Construction of random input-output codes with moderate block lengths," *IEEE Trans. Commun.*, vol. 64, no. 5, May 2016.
- [3] R. L. Rivest and A. Shamir, "How to reuse a write-once memory," *Inf. Control*, vol. 55, nos. 1-3, pp. 1-19, Dec. 1982.
- [4] R. L. Graham and N. J. A. Sloane, "Lower bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 37-43, 1980.
- [5] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, "A new table of constant weight codes," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1334-1380, 1990.
- [6] E. Agrell, A. Vardy, and K. Zeger, "Upper bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2373-2395, 2000.